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Investigation of the Exceptional Modes: $n=\pm 1, \pm 2$.

by

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ABSTRACT

It was shown (in Research Report No. 170-3) that the function

$$W_{2,n}(v) = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + \sqrt{-\frac{1}{v^2}\frac{I_n'(v)}{I_n(v)}\frac{K_n'(v)}{K_n(v)}}$$

was associated with non-attenuated modes of an idealized helical waveguide. The roots of the equation, $W_{2,n}$ = constant, determine the phase velocity of these modes. The function $W_{2,n}$ was fairly well characterized in terms of its parameters α and n for n = 0, $n \geq 3$. In this note the function $W_{2,n}$ is studied for n = 1, $\alpha \leq \frac{1}{2}$, n = 2, $\alpha \leq 1$. It has been proved that a value $v = v_1$ can be found such that $W_{2,n}$, n = 1, 2, is monotonic increasing for $v > v_1$.

This note is an addendum, to be read in conjunction with Section 9 of the earlier Report No. 170-3.

1. Introduction

In Research Report No. 170-3, A Helical Wave Guide II, by Ralph S. Phillips and Henry Malin, information was obtained concerning the non-attenuated modes that can be propagated in an idealized helical wave guide. The theory there developed left some exceptional cases to be examined. The results that were obtained came from the investigation of several groups of functions, including the following which will be used in the present paper. References in square brackets will identify these formulas in the former paper.

$$\left.\begin{aligned} W_{1,n} &= \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + \sqrt{x_n y_n} && [4.2] \\ W_{2,n} &= -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + \sqrt{x_n y_n} && [4.2] \\ x_n &= \frac{1}{v}\,\frac{I_n'(v)}{I_n(v)}, \quad y_n = -\frac{1}{v}\,\frac{K_n'(v)}{K_n(v)} && [4.3] \end{aligned}\right\} \quad (1.1)$$

$$Y_{2,n} = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + y_n \qquad [4.7] \qquad (1.2)$$

$$X_{2,n} = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + x_n \qquad [4.6] \qquad (1.3)$$

The following facts were brought out in the above paper: (i) if $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$, $n \geq 3$, $W_{2,n}$ is a monotonic increasing function [Theorem 13] ; (ii) if $\sqrt[4]{n(n-1)^2(n-2)} < \alpha < n$, $Y_{2,n}$ has a single minimum, and thereafter approaches the x-axis from below [Theorem 8] ; (iii) if $\alpha < n$, $X_{2,n}$ is negative and monotonic increasing [Theorem 6] ;

$$\left.\begin{aligned} &\text{(iv)}\quad \frac{n}{v^2}\sqrt{1+\frac{v^2}{n(n+1)}} \leq x_n \leq \frac{n}{v^2}\sqrt{1+\frac{v^2}{n^2}}, && \text{[Corollary p. 32]} \\ &\text{(v)}\quad \frac{n}{v^2}\sqrt{1+\frac{v^2}{n^2}} < y_n < \frac{n}{v^2}\sqrt{1+\frac{v^2}{n(n-1)}}, && \text{[Corollary 1, p. 46]}. \\ &\text{(vi)}\quad W_{1,n} \text{ is a positive monotonic decreasing function}. && \text{[Theorem 10]} \end{aligned}\right\} \quad (1.4)$$

It is seen that there is no information about the functions $W_{2,n}$ for $n = \pm 1, \pm 2$, which will tell us what solutions of (4.1) exist for these values of n. In the former paper some results were obtained for these cases for special values of α by graphical methods. It is the purpose of the present paper to analyze these exceptional cases further in order that the gaps left by the earlier discussion can be partially filled.

2. The Function $W_{2,1}$

The function $W_{2,1}$ [Eq. 9.17, p. 68] is initially plus infinity and for $\alpha < 1$ approaches the v-axis from below. We have not been successful in showing that $W_{2,1}$ has one and only one minimum value. We have proved the following partial description of $W_{2,1}$ which we state as a Theorem: $W_{2,1}$ is monotonic increasing for $v > v_1$ and $\alpha \leq \frac{1}{2}$ where v_1 is the minimum of $Y_{2,1}(v, \frac{1}{2})$.

Denoting $\varphi_1(v, \alpha)$ [4.10] by $\varphi_\alpha(v)$, we have

$$\varphi_\alpha(v) = \frac{1}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} \quad , \quad r = \sqrt{\frac{x_1}{y_1}} \tag{2.1}$$

and we must show that

$$W'_{2,1} = -\varphi'_{1/2} + \frac{1}{2}\sqrt{\frac{x_1}{y_1}}\; y'_1 + \frac{1}{2}\,\frac{x'_1}{\sqrt{\frac{x_1}{y_1}}} > 0 \text{ for } v > v_1, \tag{2.2}$$

or that

$$-\varphi'_{1/2} + \frac{1}{2}\, r\, y'_1 + \frac{1}{2}\,\frac{x'_1}{r} > 0$$

which will be proved if

$$r^2\,(-y'_1) - 2r\,(-\varphi'_{1/2}) + (-x'_1) < 0. \tag{2.3}$$

However, $Y'_{2,1} > 0$ for $v > v_1$. That is

$$Y_{2,1} = -\varphi_{1/2} + y_1$$

$$Y'_{2,1} = -\varphi'_{1/2} + y'_1 > 0,$$

or

$$-\varphi'_{1/2} > -y'_1 \ . \qquad (2.4)$$

Again we know that $X'_{2,1} > 0$ for $\alpha \leq 1$, which implies that

$$-\varphi'_1 > -x'_1 \ . \qquad (2.5)$$

Using inequalities (2.4) and (2.5) in (2.3) we must prove that

$$r^2 (-\varphi'_{1/2}) - 2r(-\varphi'_{1/2}) + (-\varphi'_1) < 0. \qquad (2.6)$$

It is known however that

$$x_1 < y_1 \ , \ x_1 > \varphi_{\sqrt{2}} \ , \ y_1 < \varphi_{1/\sqrt{2.1}} \ . \qquad (2.7)$$

The last inequality follows from the fact that for $v = v_1, \alpha = \frac{1}{\sqrt{2.1}}$, $Y_{2,1}$ is negative and is negative thereafter [Theorem 8, p. 34].

Inequalities (2.7) imply that

$$\sqrt[4]{\frac{1 + \frac{v^2}{2}}{1 + 2.1\, v^2}} < r < 1 \ . \qquad (2.8)$$

The left hand side of the inequality (2.6) is a quadratic function of r with the first term positive. If the inequality is true for the extreme values given in (2.8), it follows for all intermediate values. The minimum value of

$$\sqrt[4]{\frac{1 + \frac{v^2}{2}}{1 + 2.1\, v^2}} \text{ is } \sqrt{\frac{\frac{1}{2}}{2.1}} = .70 \ .$$

Hence it must be shown that

$$.49 \left[-\varphi'_{1/2}\right] - 1.40 \left[-\varphi'_{1/2}\right] - \varphi'_1 < 0 \qquad (2.9)$$

or

$$\left[-\varphi'_1\right] < .91 \left[-\varphi'_{1/2}\right] . \qquad (2.10)$$

However,

$$\varphi'_{\alpha} = - \frac{1}{v^3\sqrt{1 + \frac{v^2}{\alpha^2}}} \left[2 + \frac{v^2}{\alpha^2}\right] . \tag{2.11}$$

Substituting in (2.10) the values for φ'_1 , $\varphi'_{1/2}$, we must show that

$$\frac{2}{v^3\sqrt{1 + v^2}} (2 + v^2) < .91 \frac{2}{v^3\sqrt{1 + 4v^2}} (2 + 4v^2) \tag{2.12}$$

or that

$$(2 + v^2)^2 (1 + 4v^2) < (.91)^2 \; 4 (1 + 2v^2)^2 (1 + v^2)$$

which reduces to

$$0 < -.7 + 3.4 v^2 + 9.6 v^4 + 9.3 v^6 . \tag{2.13}$$

This will be true if

$$3.4 v^2 > .7$$

$$v > \sqrt{\frac{.7}{3.4}} = .45 .$$

The minimum of the function $Y_{21} (v, \frac{1}{2})$ occurs for a value of v somewhere between .7 and .8 . Since inequality (2.6) is clearly true for r = 1 the theorem is established.

The result of the previous theorem - in so far as the range of v is concerned - is capable of being extended by brute force. That is to say suppose we want the theorem to hold for $v \geq .5$. We know the theorem is true for $v > v_1 = 7.5$. Hence it must be proved for $.5 \leq v \leq v_1$. Now $-y_1'$ is a positive decreasing function, if it be replaced by its maximum value in the inequality (2.3) and if this inequality is established, the extension will be proved. This has actually been done but the details will not be given.

3. The Function $W_{2,2}$

It was thought that the function $W_{2,2}$ is monotone increasing, at least for a certain range of the parameter α. However,

$$W_{2,2} = \frac{1}{3} - \frac{1}{\alpha^2} + \frac{v^2}{2} \log \frac{1}{2} v + \ldots \qquad (3.1)$$

for v sufficiently small. This development about the origin shows that $W_{2,2}$ has a negative slope for small v no matter what value is assigned to α. Thus if $W_{2,2}$ starts out negative, it has a minimum (at least one) and then approaches the v-axis from below.

If $\alpha = 1$, and $v = v_1$ represents the abscissa which makes $Y_{2,2}$ a minimum, we will establish the following <u>Theorem:</u> $W_{2,2}$ <u>is a monotonic increasing function of v for</u> $v > v_1$ <u>and</u> $\alpha \leq 1$.

Setting

$$\varphi_\alpha = + \frac{2}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}}, \quad r = \sqrt{\frac{x_2}{y_2}} \qquad (3.2)$$

we must show, as in the preceding theorem that

$$r^2(-\varphi_1') - 2r(-\varphi_1') + (-\varphi_2') < 0 \qquad (3.3)$$

for all r in the range $\sqrt[4]{\dfrac{1 + \frac{v^2}{6}}{1 + \frac{v^2}{2}}} < r < 1 \qquad (3.4)$

since it is known by 1.4

$$x_2 > \sqrt{1 + \frac{v^2}{6}}, \quad y_2 < \sqrt{1 + \frac{v^2}{2}} \quad . \qquad (3.5)$$

The minimum of $\sqrt[4]{\dfrac{1 + \frac{v^2}{6}}{1 + \frac{v^2}{2}}}$ is $\sqrt[4]{\frac{1}{3}}$. If

$$\beta = 1 - r = \frac{1 - r^4}{(1 + r)(1 + r^2)} \qquad (3.6)$$

we see that since

$$(1 + r)(1 + r^2) \geq (1 + \sqrt[4]{\tfrac{1}{3}})(1 + \sqrt{\tfrac{1}{3}})$$

$$> \frac{\frac{2}{3}}{1 - \sqrt[4]{\frac{1}{3}}} = \frac{.660}{.240} \geq 2.77,$$

$$\beta < \frac{1 - r^4}{(2.77)} \quad . \tag{3.7}$$

Using (3.4) we see that

$$\beta < \frac{2v^2}{(2.77)(2)(3)} \cdot \frac{1}{1 + \frac{v^2}{2}} \quad . \tag{3.8}$$

But

$$\left(1 + \frac{v^2}{2}\right)^2 > \left(1 + \frac{v^2}{4}\right)\left(1 + \frac{v^2}{2}\right) \quad , \tag{3.9}$$

thus

$$\beta^2 < \frac{v^4}{(6.9)(1 + \frac{v^2}{2})(1 + \frac{v^2}{4})} \tag{3.10}$$

and so

$$1 - 2\beta^2 > 1 - \frac{2v^4}{6.9\,(1 + \frac{v^2}{2})(1 + \frac{v^2}{4})} \quad . \tag{3.11}$$

Returning now to inequality (3.3), we may write it as

$$(2 - r)\, r\, (-\varphi_1') > -\varphi_2' \tag{3.12}$$

which in terms of β becomes

$$(1 - \beta^2)(-\varphi_1') > -\varphi_2' \quad . \tag{3.13}$$

Since

$$(1-\beta^2)^2 > 1-2\beta^2, \quad 0 < \beta < 1 - r < 1,$$

Inequality (3.13) will be true if

$$(1 - 2\beta^2)(-\varphi_1')^2 > (-\varphi_2')^2 \tag{3.14}$$

Employing (3.11) and differentiating (3.2) we must show that

$$\left(1 - \frac{v^4}{27(1+\frac{v^2}{4})(1+\frac{v^2}{2})}\right)(1+\frac{v^2}{4})(4+4v^2+v^4)$$

$$> (1+v^2)(4+v^2+\frac{v^4}{16}) \tag{3.15}$$

or that

$$4 + 4v^2 + v^4 + v^2 + v^4 + \frac{v^6}{4} - \frac{v^4}{27}(1+\frac{v^2}{2})$$

$$> 4 + v^2 + \frac{v^4}{16} + 4v^2 + v^4 + \frac{v^6}{16} \tag{3.16}$$

Inequality (3.16) is true if like inequalities are true for the coefficients of like powers.

For v^4 we must show that

$$1 - \frac{4}{27} > \frac{1}{16}$$

$\frac{23}{27} > \frac{1}{16}$ which is all right.

For v^6 we want

$$\frac{1}{4} - \frac{2}{27} > \frac{1}{16}$$

$$.250 - .074 = .176 > .06$$

Hence the theorem is proved.

4. Conclusions

The non-attenuated modes of the helical wave guide in which we are at present interested are gotten from the real solutions of the equation

$$\frac{\delta}{\alpha} = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \pm \sqrt{x_n y_n} \qquad n = \pm 1, \pm 2 \qquad (4.1)$$

The left-hand side of this equation is a positive constant; the right-hand side can be written in terms of the functions $W_{1,n}$, $W_{2,n}$.

$$\frac{\delta}{\alpha} = W_{2,n} \qquad n = 1,2 \qquad (4.2)$$

$$\frac{\delta}{\alpha} = W_{1,|n|} \qquad n = -1, -2 \qquad (4.3)$$

and

$$\frac{\delta}{\alpha} = -W_{1,n} \qquad n = 1,2 \qquad (4.4)$$

$$= -W_{2,|n|} \qquad n = -1, -2 \qquad (4.5)$$

In so far as Eqs. (4.3) and (4.4) are concerned the question of real solutions of Eq. (4.1) is readily settled. Since $W_{1,n}$ is a positive monotonic decreasing function of α, Eq. (4.3), for any positive δ/α has one and only one real solution; Eq. (4.4) has no real solution. As regards the remaining two equations we shall discuss them only from the knowledge gained from the two theorems which have been established i.e., $W_{2,1}$, $W_{2,2}$ are negative monotonic increasing functions for $v > v_1$ $\alpha \leq \frac{1}{2}$, $n = 1$; $v > v_2$, $\alpha \leq 1$, $n = 2$. It is seen that equation (4.2) has no solution for $n = 1,2$ which is greater than v_1. Finally Eq. (4.5) has one and only one solution greater than v_1 $(n = 1)$ and one and only one solution greater than v_2 $(n = 2)$ providing δ/α is not greater than $-W_{2,1}(v_1)$ or $-W_{2,2}(v_2)$. In conclusion the authors believe, although they cannot prove, that the functions $W_{2,n}$ $(n = 1,2)$ have single minima and thereafter increase for $\alpha \leq 1$, $\alpha \leq 2$.